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On the AKSZ formulation of the Rozansky-Witten theory and beyond

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Abstract

Using the AKSZ formalism, we construct the Batalin-Vilkovisky master action for the Rozansky-Witten model, which can be defined for any complex manifold with a closed $(2,0)$-form. We also construct the holomorphic version of Rozansky-Witten theory defined over Calabi-Yau 3-fold.

1 Introduction

In [10] L. Rozansky and E. Witten introduced a 3-dimensional topological sigma model whose target space is (compact or asymptotically flat) hyperKähler manifold M. We refer to this model as the Rozansky-Witten (RW) model. The Feynman diagram expansion of the partition function for this model gives rise to the finite type invariant of 3-manifolds (Rozansky-Witten invariants) similar to those appearing in perturbative Chern-Simons theory. The reader may consult [11] for the review of the RW invariants and [6] for the recent development in the RW-theory.

In this work we discuss the Batalin-Vylkovisky (BV) formulation of the RW model. Using the Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) prescription [1] we construct master action for 3-dimensional topological sigma model which upon gauge fixing coincides with RW model. The AKSZ-BV framework is conceptually powerful and it allows to address certain issues independently from a particular gauge fixing. Moreover the whole exercise is interesting on its own since it provides the relatively exotic example of the AKSZ construction. This paper was inspired by the remarks from [7].

The paper is organized as follows. In Section 2 we remind the basic aspects of AKSZ formalism. Section 3 contains the AKSZ construction of 3-dimensional sigma model with hyperKähler target. In Section 4 we demonstrate that upon the specific gauge fixing the BV model corresponds to the original formulation of RW model from [10]. As a simple corollary of the AKSZ formalism in Section 5 we present the holomorphic version of RW theory defined over a Calabi-Yau 3-fold. Section 6 contains the summary.

2 AKSZ formalism

In this Section we review the AKSZ construction [1] of the solutions of the classical master equation within BV formalism. We closely follow the presentation given in [9] and we use the language of graded manifolds which are sheaves of $\mathbb{Z}$-graded commutative algebras over a smooth manifold, for further details the reader may consult [12]. We consider both the real and complex cases and treat them formally on equal footing. However in complex case the additional care is required (see [1] for further details).

The AKSZ solution of the classical master equation is defined starting from the following data:

The source: A graded manifold $\mathcal{N}$ endowed with a homological vector field D and a measure $\int_{\mathcal{N}} \mu$ of degree $-n-1$ for some positive integer n such that the measure is invariant under D.

The target: A graded symplectic manifold $(\mathcal{M}, \omega)$ with $\deg(\omega) = n$ and a homological vector field Q preserving ω. We require that Q is Hamiltonian, i.e. there exists a function Θ of degree $n+1$ such that $Q = \{\Theta, -\}$. Therefore Θ satisfies the following Maurer-Cartan equation

$$\{\Theta, \Theta\} = 0 \ .$$

Introduce the space of maps between $\mathcal{N}$ and $\mathcal{M}$, $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ which should be understood as the space of morphisms of sheaves, which is again a sheaf over $C^\infty(\Sigma, M)$, where Σ and M are the underlying ordinary smooth manifolds of $\mathcal{N}$ and $\mathcal{M}$ respectively. The space $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ is naturally equipped with the odd symplectic structure. Moreover D and Q can be interpreted as homological vector fields acting on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ and preserving this odd symplectic structure. The AKSZ solution S_{BV} is the Hamiltonian for homological vector field $D+Q$ on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ and thus it satisfies automatically the classical master equation.

Let us provide some details for this elegant construction. Pick a map $\Phi \in \mathrm{Maps}(\mathcal{N}, \mathcal{M})$. We choose a set of coordinates $X^A = \{x^\mu; \psi^m\}$ on the target $\mathcal{M}$, where $\{x^\mu\}$ are the coordinates for an open $U \subset M$ and $\{\psi^m\}$ are the coordinates in the formal directions. We also choose the coordinates $\{\xi^\alpha; \theta^a\}$ on the source $\mathcal{N}$, where $\{\xi^\alpha\}$ are the local coordinates on Σ and $\{\theta^a\}$ are the coordinates in the formal directions of $\mathcal{N}$. The superfield Φ is defined as an expansion over the formal coordinates of $\mathcal{N}$ for $\Phi_0^{-1}(U)$

$$\Phi^A = \Phi_0^A(u) + \theta^a \Phi_a^A(u) + \frac{1}{2}\theta^{a_2}\theta^{a_1}\Phi^A_{a_1 a_2}(u) + \dots \ . \tag{1}$$

The symplectic form ω of degree n on $\mathcal{M}$ can be written in the Darboux coordinates $\omega = dX^A \omega_{AB} dX^B$. Using this form we define the symplectic form of degree -1 on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ as

$$\omega_{BV} = \int\limits_{\mathcal{N}} \mu \ \ \delta\Phi^A \ \omega_{AB} \ \delta\Phi^B \ . \tag{2}$$

Thus the space of maps $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ is naturally equipped with the odd Poisson bracket $\{\ ,\ \}$. Since the space $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ is infinite dimensional we cannot define the BV Laplacian properly. We can only talk about the naive Laplacian adapted to the local field-antifield splitting. However on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ we can discuss the solutions of the classical master equation. The AKSZ action then reads

$$S_{BV}[\Phi] = S_{kin}[\Phi] + S_{int}[\Phi] = \int\limits_{\mathcal{N}} \mu \ \left(\frac{1}{2}\Phi^A \omega_{AB} D\Phi^B + (-1)^{n+1}\Phi^*(\Theta)\right) \tag{3}$$

and it solves the classical master equation $\{S_{BV}, S_{BV}\} = 0$ with respect to the bracket defined by the symplectic structure (2). In writing the solution we did not have to use the

Darboux coordinate on $\mathcal{M}$. Assuming that ω admits Liouville form Ξ the first term in (3) can be written

$$S_{kin}[\Phi] = \int\limits_{\mathcal{N}} \mu\ \Xi_A(\Phi) D\Phi^A\ . \tag{4}$$

Since the measure μ is invariant under D, S_{kin} depends[1] only on ω, not a concrete choice of Ξ. The action (3) is invariant under all orientation preserving diffeomorphisms of Σ and thus defines a topological field theory. The solutions of the classical field equations of (3) are graded differentiable maps $(\mathcal{N}, D) \to (\mathcal{M}, Q)$, i.e. maps which commute with the homological vector fields.

The homological vector field Q on $\mathcal{M}$ defines a complex on $C^\infty(\mathcal{M})$ whose cohomology we denote $H_Q(\mathcal{M})$. Take $f \in C^\infty(\mathcal{M})$ and expand $\Phi^* f$ in the formal variables on $\mathcal{N}$

$$\Phi^* f = O^{(0)}(f) + \theta^a O^{(1)}_a(f) + \frac{1}{2}\theta^{a_2}\theta^{a_1} O^{(2)}_{a_1 a_2}(f) + \dots\ .$$

We denote by δ_{BV} the Hamiltonian vector field for S_{BV}, which is homological as a consequence of classical master equation. The action of δ_{BV} on $\Phi^* f$ is given by the following expression

$$\delta_{BV}(\Phi^* f) = \{S_{BV}, \Phi^* f\} = D\Phi^* f + \Phi^* Q f\ .$$

Thus if $Qf = 0$ and μ_k is a D-invariant linear functional on the functions of $\mathcal{N}$ (e.g., a representative of an homology class of Σ), then $\mu_k(O^{(k)}(f))$ is δ_{BV}-closed and can serve as a classical observable. Therefore $H_Q(\mathcal{M})$ naturally defines a set of classical observables in the theory. The classical action (3) can be deformed to the first order by

$$\int\limits_{\mathcal{N}} \mu\ O^{(n+1)}(f)$$

with $f \in H_Q(\mathcal{M})$.

The standard choice for the source is odd tangent bundle $\mathcal{N} = T[1]\Sigma$, for any smooth manifold Σ of dimension $n+1$, with $D = d$ the de Rham differential over Σ and the canonical coordinate measure.

Example 1 (**Chern-Simons theory**) The Chern-Simons model is easily constructed within AKSZ framework [1]. The source $\mathcal{N} = T[1]\Sigma_3$ with de Rham differential and canonical integration. The target is $\mathcal{M} = \mathbf{g}[1]$ where $\mathbf{g}$ is a metric Lie algebra.

[1]Here we consider the case when $\partial\Sigma = \emptyset$. For the case with a boundary the reader may consult [3].

Example 2 (**Poisson sigma model**) The AKSZ approach was applied to 2-dimensional Poisson sigma model in [3]. In this case the source $\mathcal{N}$ is odd tangent $T[1]\Sigma_2$ for 2-dimensional manifold Σ_2 equipped with de Rham differential and canonical integration measure. The target $\mathcal{M}$ is odd cotangent bundle $T^*[1]M$ with homological vector field associated to Poisson structure.

Example 3 (**Courant sigma model**) The AKSZ construction can be applied to 3-dimensional Courant sigma model which associates for Courant algebroid a 3-dimensional topological field theory, [5], [9]. The simplest example of this model corresponds to the following choice of $\mathcal{N} = T[1]\Sigma_3$ and $\mathcal{M} = T^*[2]T^*[1]M$.

There exists other choices for the source supermanifold. For example, if Σ is complex manifold with the holomorphic volume form then $\mathcal{N} = T^{(1,0)}\Sigma$ is equipped with integration and it is invariant under the homological vector field $D = \bar{\partial}$, the Dolbeault differential.

Example 4 (**holomorphic Chern-Simons theory**) The source is $\mathcal{N} = T^{(1,0)}[1]\Sigma_6$ where Σ_6 is Calabi-Yau 3-fold. The measure has degree -3 and thus the target $\mathcal{M} = \mathbf{g}[1]$ with $\mathbf{g}$ being a metric Lie algebra will work. It is simple exercise to check that the resulting theory would be the BV action for the holomorphic Chern-Simons theory [13].

The AKSZ prescription is algebraic in its nature and thus it can be generalized even further, see for example [2].

3 Rozansky-Witten model from AKSZ

We are interested in the construction of 3-dimensional topological sigma model. If we choose $T[1]\Sigma_3$ as a source manifold then the target $\mathcal{M}$ should be graded symplectic manifold of degree 2. As in [9] we consider the even symplectic manifold $T^*[2]T^*[1]M$ with the symplectic structure

$$\omega = \delta P_\mu \wedge \delta X^\mu + \delta v^\mu \wedge \delta q_\mu \ , \tag{5}$$

where the following allocation of degrees is assumed $\deg(X) = 0$, $\deg(P) = 2$, $\deg(v) = \deg(q) = 1$. If we assume that M is a complex manifold then there exists a homological vector field Q of degree 1

$$Q = P_{\bar{i}} \frac{\partial}{\partial q_{\bar{i}}} + v^{\bar{i}} \frac{\partial}{\partial X^{\bar{i}}} \ , \tag{6}$$

where we use the complex coordinates $(i, \bar{i})$ on M. Q preserves ω and the corresponding Hamiltonian is $\Theta = P_{\bar{i}} v^{\bar{i}}$. Picking up the local coordinates X, P, v, q on $T^*[2]T^*[1]M$, the space of maps

$$T[1]\Sigma_3 \longrightarrow T^*[2]T^*[1]M$$

can be described by set of superfields $\boldsymbol{X}$, $\boldsymbol{P}$, $\boldsymbol{v}$, $\boldsymbol{q}$. For the components of $\boldsymbol{X}$ we use the following conventions

$$\boldsymbol{X}^\mu = X^\mu + \theta^a X^\mu_a + \frac{1}{2!}\theta^b\theta^a X^\mu_{ab} + \frac{1}{3!}\theta^c\theta^b\theta^a X^\mu_{abc} \tag{7}$$

and the same conventions for other superfields. The space of maps is equipped with the odd symplectic structure

$$\omega_{BV} = \int d^3\theta d^3\xi \ \left(\delta \boldsymbol{P}_\mu \wedge \delta \boldsymbol{X}^\mu + \delta \boldsymbol{v}^\mu \wedge \delta \boldsymbol{q}_\mu\right) \ . \tag{8}$$

Applying the AKSZ construction to this case we arrive on the following action

$$S_{BV} = \int d^3\theta d^3\xi \left(\boldsymbol{P}_\mu D\boldsymbol{X}^\mu + \boldsymbol{q}_\mu D\boldsymbol{v}^\mu + \boldsymbol{P}_{\bar{i}} \boldsymbol{v}^{\bar{i}}\right) \ , \tag{9}$$

which automatically satisfies the classical master equation. This model is an example of family of 3-dimensional sigma models associated to the generalized complex manifold (for further details see [4]). However here we are interested in an exotic modification of this model.

Assume that M is complex manifold with a closed $(2,0)$-form Ω. In this case the symplectic structure (5) on $T^*[2]T^*[1]M$ can be modified as follows

$$\omega = \delta P_\mu \wedge \delta X^\mu + \delta v^\mu \wedge \delta q_\mu + \Omega_{ij}\delta X^i \wedge \delta X^j \ , \tag{10}$$

where we should require[2] that $\deg(\Omega) = 2$. In other words we can think about introducing the formal parameter of degree 2 and putting it in front of Ω. Strictly speaking $T^*[2]T^*[1]M$ with (10) is not a graded symplectic manifold since we have an auxiliary parameter with non-zero degree. However the AKSZ construction still goes through. The homological vector field Q in (6) preserves the new symplectic structure (10) and the corresponding Hamiltonian is $P_{\bar{i}} v^{\bar{i}}$. On the space of maps $T[1]\Sigma_3 \longrightarrow T^*[2]T^*[1]M$ the following odd symplectic structure is defined

$$\omega_{BV} = \int d^3\theta d^3\xi \ \left(\delta \boldsymbol{P}_\mu \wedge \delta \boldsymbol{X}^\mu + \delta \boldsymbol{v}^\mu \wedge \delta \boldsymbol{q}_\mu + \Omega_{ij}\delta \boldsymbol{X}^i \wedge \delta \boldsymbol{X}^j\right) \ . \tag{11}$$

[2] Alternatively we can work only within $\mathbb{Z}_2$-grading.

Writing locally $2\Omega_{ij} = \partial_i \xi_j - \partial_j \xi_i$ with ξ being $(1,0)$ holomorphic form we can apply the AKSZ construction and arrive at the following BV action

$$S_{BV} = \int d^3\theta d^3\xi \left(\boldsymbol{P}_\mu D\boldsymbol{X}^\mu + \boldsymbol{q}_\mu D\boldsymbol{v}^\mu + \xi_i(\boldsymbol{X}) D\boldsymbol{X}^i + \boldsymbol{P}_{\bar{i}} \boldsymbol{v}^{\bar{i}} \right) \ . \tag{12}$$

Despite its appearance the action (12) depends only on Ω (not ξ) in the case $\partial\Sigma_3 = \emptyset$. The direct calculation shows

$$\{S_{BV}, S_{BV}\} = \int d^3\theta d^3\xi \ \left(D(\boldsymbol{P}_\mu D\boldsymbol{X}^\mu + \boldsymbol{q}_\mu D\boldsymbol{v}^\mu + \boldsymbol{P}_{\bar{i}} \boldsymbol{v}^{\bar{i}}) + \Omega_{ij} D\boldsymbol{X}^i D\boldsymbol{X}^j \right) \ .$$

The first term is immediately seen to be a surface term. To see that the second term is also a surface term, we have to perform the θ integrals

$$\int d^3\theta d^3\xi \ \Omega_{ij} D\boldsymbol{X}^i D\boldsymbol{X}^j = \int d^3\xi \ \epsilon^{abc} \partial_c (\Omega_{ij} (\partial_a X^i) X^j_b) \tag{13}$$

and use $d\Omega = 0$.

The model defined by (8) and (9) is formally related to the model defined by (11) and (12) through the formal shift $\boldsymbol{P}_i \to \boldsymbol{P}_i + \xi_i$ with $\deg(\xi) = 2$ and ξ being a holomorphic $(1,0)$ form such that $\Omega = d\xi$. Indeed there exists a whole family of models where Ω-term in (11) and ξ-term in (12) enter with the different numerical coefficients, which still satisfy the classical master equation.

4 Gauge fixing of Rozansky-Witten model

In the previous section we constructed the classical BV action for 3-dimensional topological sigma model with the target being a complex manifold M admitting the closed $(2,0)$ form Ω. In this section we discuss the gauge fixing of this model. In particular we show that when M is hyperKähler the gauged fixed version of (12) is exactly the Rozansky-Witten model [10].

Let M be hyperKähler manifold with metric g and holomorphic symplectic form Ω which is covariantly constant with respect to Levi-Civita connection. The gauge fixing in BV formalism consists of evaluating the BV action on the Lagrangian manifold. The main complication is related to the properties of $T^*[2]T^*[1]M$ (e.g. see [8]). $T^*[2]T^*[1]M$ is a vector bundle over vector bundle and thus P transforms in non-tensorial fashion under the change of coordinates on M. Moreover many components of superfields $\boldsymbol{X}$, $\boldsymbol{P}$, $\boldsymbol{q}$ and $\boldsymbol{v}$ transform in rather complicated way. The way out is the introduction of the connection and redefining some operations in the covariant way. Let $\Gamma^\nu_{\mu\rho}$ be the Levi-Civita connection for

Kähler metric g. We redefine the coordinate of degree 2 on $T^*[2]T^*[1]M$ and correspondingly the superfiled $\boldsymbol{P}$ as follows

$$\mathbb{P}_\mu = \boldsymbol{P}_\mu + \Gamma^\nu_{\mu\rho}\boldsymbol{q}_\nu \boldsymbol{v}^\rho \ . \tag{14}$$

The master action (12) becomes

$$S_{BV} = \int d^3\theta d^3\xi \left(\mathbb{P}_\mu D\boldsymbol{X}^\mu + \boldsymbol{q}_\mu \nabla_D \boldsymbol{v}^\mu + \xi_i(\boldsymbol{X})D\boldsymbol{X}^i + \mathbb{P}_{\bar{i}}\boldsymbol{v}^{\bar{i}}\right) \ , \tag{15}$$

where

$$\nabla_D \boldsymbol{v}^\mu = D\boldsymbol{v}^\mu + \Gamma^\mu_{\nu\rho} D\boldsymbol{X}^\nu \boldsymbol{v}^\rho \ . \tag{16}$$

In writing (15) we used the properties of the Levi-Civita connection for Kähler metric. We can also covariantize the θ-derivatives

$$\nabla_{\theta^a} := \delta^\mu_\rho \partial_{\theta^a} + \Gamma^\mu_{\nu\rho}(\partial_{\theta^a}\boldsymbol{X}^\nu) \tag{17}$$

and define the covariant components of the superfields. For example, we define

$$X_{ab} := \nabla_{\theta^a}\partial_{\theta^b}\boldsymbol{X}| \ , \tag{18}$$

where the vertical bar | denotes "the $\theta = 0$ part". However we have to keep in mind that now ∇_{θ^a} and ∇_{θ^b} do not anticommute. The odd symplectic form (11) can be rewritten in the covariantized superfields as follows

$$\omega_{BV} = \int d^3\theta d^3\xi \ \left(\delta\left(\mathbb{P}_\mu \delta\boldsymbol{X}^\mu + \boldsymbol{q}_\mu \nabla_\delta \boldsymbol{v}^\mu\right) + \Omega_{ij}\delta\boldsymbol{X}^i \wedge \delta\boldsymbol{X}^j)\right) \ , \tag{19}$$

where

$$\nabla_\delta \boldsymbol{v}^\mu = \delta\boldsymbol{v}^\mu + \Gamma^\mu_{\nu\rho}\delta\boldsymbol{X}^\nu \boldsymbol{v}^\rho \ . \tag{20}$$

The θ-integration of covariant scalar expression is defined as

$$\int d^3\theta ... = \frac{1}{6}\int \epsilon_{abc} d\theta^a d\theta^b d\theta^c ... = \frac{1}{6}\int \epsilon_{abc}\partial_{\theta^a}\partial_{\theta^b}\partial_{\theta^c}...| = \frac{1}{6}\int \epsilon_{abc}\nabla_{\theta^a}\nabla_{\theta^b}\nabla_{\theta^c}...| \tag{21}$$

Now equipped with these tools we perform the gauge fixing by choosing the suitable Lagrangian submanifold for (19) and evaluating the action (15) on it.

We now expand the symplectic form (19) in components and we shall ignore the $\boldsymbol{q}, \boldsymbol{v}$ sector

$$\begin{aligned}
\omega_{BV} &= \frac{1}{6}\int \epsilon^{abc} d\theta^a d\theta^b \delta\left(\partial_{\theta^c}\left(\mathbb{P}_\mu \delta\boldsymbol{X}^\mu\right) + 2\Omega_{ij}(\partial_{\theta^c}\boldsymbol{X}^i)\delta\boldsymbol{X}^j + ...\right) \\
&= \frac{1}{6}\int \epsilon^{abc}\delta\big((\mathcal{P}_{abc\mu} + 2\Omega_{i\mu}X^i_{abc} + 2\Omega_{ij}X^i_c R(X_a, {}_\mu, {}^j, X_b) - 3R(X_b, {}_\mu, \mathcal{P}_a, X_c))\delta X^\mu \\
&\quad -(-3\mathcal{P}_{ab\mu} + 6\Omega_{\mu j}X^j_{ab})\nabla_\delta X^\mu_c + 3\mathcal{P}_{c\mu}\nabla_\delta X^\mu_{ab} + \mathcal{P}_\mu(\nabla_{\theta^a}\nabla_{\theta^b}\nabla_{\theta^c}\delta\boldsymbol{X}^\mu)| + ...\big) \ ,
\end{aligned} \tag{22}$$

where R is the curvature tensor is defined

$$[\nabla_\mu, \nabla_\nu] v^\alpha := R_{\mu\nu}{}^\alpha{}_\beta v^\beta$$

and the components are

$$\mathcal{P}_\mu = \mathbb{P}_\mu| \ , \qquad \mathcal{P}_{a\mu} = \nabla_{\theta^a}\mathbb{P}_\mu| \ , \qquad \mathcal{P}_{ab\mu} = \nabla_{\theta^a}\nabla_{\theta^b}\mathbb{P}_\mu| \ , \qquad \mathcal{P}_{abc\mu} = \nabla_{\theta^a}\nabla_{\theta^b}\nabla_{\theta^c}\mathbb{P}_\mu| \ .$$

From (22) we can pick the following Lagrangian submanifold

$$\mathcal{P}_\mu = 0 \ , \qquad \mathcal{P}_{a\bar{i}} = 0 \ , \qquad \mathcal{P}_{ai} = 0 \ , \qquad \mathcal{P}_{ab\mu} = 2\Omega_{\mu j} X^j_{ab}$$

$$\mathcal{P}_{abc\mu} = -2\Omega_{i\mu} X^i_{abc} - 2\Omega(X_{a,\,j}) R(X_{b,\,\mu,}{}^{j}, X_c)$$

together with $\boldsymbol{q} = 0$ (i.e., all components of $\boldsymbol{q}$ are set to zero, which justifies why we ignored the $\boldsymbol{q}, \boldsymbol{v}$ sector from ω_{BV}). The BV action (15) written in components is

$$\begin{aligned} S_{BV} &= \frac{1}{6}\int d^3\xi \ \epsilon^{abc}\Big(3\mathcal{P}_{a\mu}\nabla_{[c}X^\mu_{b]} + 3\mathcal{P}_{ab\mu}\partial_c X^\mu + 6\Omega_{ij}(X^i_c \nabla_b X^j_a + X^i_{bc}\partial_a X^j) \\ &\quad + \mathcal{P}_{abc\bar{i}} v^{\bar{i}} + \mathcal{P}_{ab\bar{i}} v^{\bar{i}}_c + \mathcal{P}_{a\bar{i}} v^{\bar{i}}_{bc} + \mathcal{P}_{\bar{i}} v^{\bar{i}}_{abc} + \ldots\Big) \ , \end{aligned} \tag{23}$$

where dots stand for $\boldsymbol{q}_\mu \nabla_D \boldsymbol{v}^\mu$-term. Evaluating the action (23) on the above Lagrangian we recover the RW model

$$S_{RW} = \int d^3\xi \ \epsilon^{abc}\Big(\Omega(X_c, \nabla_b X_a) + \frac{1}{3} R_{k\bar{k}}{}^i{}_j X^k_b \Omega_{li} X^l_a X^j_c v^{\bar{k}}\Big)$$

or written in terms of differential forms on Σ_3

$$S_{RW} = -6\int \left(\Omega_{ij} X^i_{(1)} \wedge d^\nabla X^j_{(1)} + \frac{1}{3} R_{k\bar{k}}{}^i{}_j X^k_{(1)} \wedge \Omega_{li} X^l_{(1)} \wedge X^j_{(1)} v^{\bar{k}}\right) \ , \tag{24}$$

where the only non-zero fields left in the model are odd 1-form $X^i_{(1)} = X^i_a d\xi^a$, odd scalar $v^{\bar{i}}$ and even coordinate X^μ. The BRST transformations are obtained by restricting the BV transformation $\delta_{BV} \cdot = \{S_{BV}, \cdot\}$ to the Lagrangian submanifold

$$\begin{aligned} \{S_{BV}, \boldsymbol{X}^{\bar{i}}\} = D\boldsymbol{X}^{\bar{i}} + \boldsymbol{v}^{\bar{i}} \ &\Rightarrow \ \delta X^{\bar{i}} = v^{\bar{i}} \ , \\ \{S_{BV}, \boldsymbol{X}^i\} = D\boldsymbol{X}^{\bar{i}} \ &\Rightarrow \ \delta X^i = 0 \ \& \ \delta X^i_{(1)} = -dX^i \ , \\ \{S_{BV}, \boldsymbol{v}^{\bar{i}}\} = D\boldsymbol{v}^{\bar{i}} \ &\Rightarrow \ \delta v^{\bar{i}} = 0 \ . \end{aligned}$$

The action (24) is invariant under these BRST transformation by construction [1]. For the sake of perturbation theory, we need a nondegenerate kinetic term, this can be done through adding to the action (24) a BRST exact term

$$\begin{aligned} S^{kin}_{RW} &= -\delta\left(\left(\int d^3\xi\sqrt{h}\ h^{ab} g_{i\bar{j}} X^i_a \partial_b X^{\bar{j}}\right)\right) = \int d^3\xi\sqrt{h}\left(h^{ab} g_{i\bar{j}}\partial_a X^i \partial_b X^{\bar{j}} + h^{ab} g_{i\bar{j}} X^i_a \nabla_b v^{\bar{j}}\right) \\ &= \int g_{i\bar{j}} dX^i_{(0)} \wedge * dX^{\bar{j}}_{(0)} + g_{i\bar{j}} X^i_{(1)} \wedge * d^\nabla v^{\bar{j}} \ , \end{aligned} \tag{25}$$

where we used the metric h on Σ_3. In (24) and (25) d^∇ is the covariant version of de Rham differential, e.g.

$$d^\nabla v^\mu = dv^\mu + \Gamma^\mu_{\nu\rho} dX^\nu v^\rho \ .$$

The exact term (25) can be generated right away by a change of the Lagrangian submanifold by an appropriate gauge fixed fermion.

In this section we analyzed the gauge fixing of the BV model introduced previously. In our setup we have used the Kähler metric g and the fact that the holomorphic $(2,0)$-form Ω is covariantly constant with respect to the Levi-Civita connection. The RW action above is by construction BRST invariant, but one can nonetheless check this explicitly. For this one would use some properties of the curvature tensor, some of which are peculiar to a Kähler manifold and the covariant constancy of Ω,

$$\nabla_{[\rho} R_{\mu\nu]}{}^\beta{}_\alpha = 0;\ R_{ij}{}^\beta{}_\alpha = 0;\ R_{i\bar{i}}{}^{\bar{k}}{}_j = R_{j\bar{i}}{}^{\bar{k}}{}_i;\ R_{\mu\nu}{}^l{}_{[j}\Omega_{i]l} = 0\ .$$

This setup is realized for a hyperKähler manifold. One can give up the property of the metric being Kähler and the property of Ω being covariantly constant. In this case the analysis will be messier with a number of extra terms. Moreover, as far as BV formalism is concerned, we do not need to use anywhere that Ω is non-degenerate (i.e., it is a holomorphic symplectic structure). However the present gauge fixing will lead to degenerate kinetic term for the fermions. There may exists a different gauge giving rise to a well-defined perturbation theory and thus leading to a generalization of RW model to any complex manifold with a closed $(2,0)$-form.

5 Holomorphic Rozansky-Witten theory

From previous analysis we saw that the RW model corresponds to AKSZ construction with the source $T[1]\Sigma_3$ and the target $T^*[2]T^*[1]M$ with the formal symplectic structure (10) of degree 2, where M is complex manifold with a closed $(2,0)$-form (e.g., M can be hyperKähler). The space of maps $\mathrm{Maps}(T[1]\Sigma_3, T^*[2]T^*[1]M)$ is equipped with the symplectic form (11) of degree -1 since the source $T[1]\Sigma_3$ has a canonical measure of degree -3. Thus the whole construction will work if we replace $T[1]\Sigma_3$ by another graded manifold equipped with homological vector field D and with invariant measure of degree -3. For example, we can take $T^{0,1}[1]\Sigma_6$ where Σ_6 is a complex manifold with the holomorphic volume form. This choice of a source gives rise to the holomorphic version of RW model, very much in analogy with holomorphic Chern-Simons theory introduced in [13]. Below we sketch the construction of the holomorphic RW theory. Our construction was inspired by the comments in [7].

The source manifold is taken to be $T^{0,1}[1]\Sigma_6$ with Σ_6 being a complex 6-dimensional manifold with a holomorphic volume form Ψ (i.e., Ψ is nowhere vanishing closed $(3,0)$-form), which is written in complex coordinates $(z^a, \bar{z}^{\bar{a}})$ as

$$\Psi = \rho(z)\ dz^1 \wedge dz^2 \wedge dz^3\ ,$$

where $\rho(z)$ is a holomorphic density. The integration on $T^{0,1}[1]\Sigma_6$ is defined as

$$\int \rho\ d^3\bar{\theta} d^6\xi\ ... = \int \rho\ d^3 z\ d^3\bar{\theta} d^3\bar{z}\ ...$$

and it is of degree -3. On $T^{0,1}[1]\Sigma_6$ the homological vector field $D = \bar{\theta}^{\bar{a}}\partial_{\bar{a}}$ corresponds to the Dolbeault differential $\bar{\partial}$. If $\partial\Sigma_6 = \emptyset$ the above measure is invariant under D. Thus most of the construction of the holomorphic RW theory can be carried over from section 3 by replacing the de Rham differential with $\bar{\partial}$. Thus on $\mathrm{Maps}(T^{(0,1)}[1]\Sigma_6, T^*[2]T^*[1]M)$ the symplectic form is

$$\omega_{BV} = \int \rho\ d^3\bar{\theta} d^6\xi\ \left(\delta \boldsymbol{X}^\mu \wedge \delta \boldsymbol{P}_\mu + \delta \boldsymbol{v}^\mu \wedge \delta \boldsymbol{q}_\mu + \Omega_{ij}\delta \boldsymbol{X}^i \wedge \delta \boldsymbol{X}^j\right) \tag{26}$$

and the master action is

$$S_{BV} = \int \rho\ d^3\bar{\theta} d^6\xi \left(\boldsymbol{P}_\mu D\boldsymbol{X}^\mu + \boldsymbol{q}_\mu D\boldsymbol{v}^\mu + \xi_i D\boldsymbol{X}^i + \boldsymbol{P}_{\bar{i}}\boldsymbol{v}^{\bar{i}}\right)\ , \tag{27}$$

which satisfies the classical master equation by construction.

The gauge fixing of this model can be done in complete analogy with the real case described in section 4. Skipping the details the gauge fixed action can be written in terms of differential forms on Σ_6

$$S_{hRW} \ = \ -6\int \Psi \wedge \left(\Omega_{ij} X^i_{(0,1)} \wedge \bar{\partial}^\nabla X^j_{(0,1)} + \frac{1}{3} R_{k\bar{k}}{}^i{}_j X^k_{(0,1)} \wedge \Omega_{li} X^l_{(0,1)} \wedge X^j_{(0,1)} v^{\bar{k}}\right)\ , \tag{28}$$

where the only non-zero fields left in the model are odd $(0,1)$-form $X^i_{(0,1)} = X^i_{\bar{a}} dz^{\bar{a}}$, odd scalar $v^{\bar{i}}$ and even coordinate X^μ. The BRST transformations are obtained by restricting the BV transformation $\delta_{BV}\cdot = \{S_{BV}, \cdot\}$ to the Lagrangian submanifold

$$\delta X^{\bar{i}} = v^{\bar{i}}\ , \qquad \delta X^i = 0\ , \qquad \delta X^i_{(0,1)} = -\bar{\partial} X^i\ , \qquad \delta v^{\bar{i}} = 0\ .$$

In order to have a well-defined perturbation theory we add to S_{hRW} the BSRT-exact kinetic term

$$S^{kin}_{hRW} \ = \ -\delta\left(\int d^6\xi \sqrt{h} h^{\bar{a}b} g_{i\bar{j}} X^i_{\bar{a}} \partial_b X^{\bar{j}}\right) = \int d^6\xi \sqrt{h}\left(h^{\bar{a}b} g_{i\bar{j}} \bar{\partial}_{\bar{a}} X^i \partial_b X^{\bar{j}} + h^{\bar{a}b} g_{i\bar{j}} X^i_{\bar{a}} \nabla_b v^{\bar{j}}\right)\ ,$$

where h is a Hermitian metric on Σ_6. Indeed in order to have a well-defined kinetic term we have to require that h is a Kähler metric[3] and thus Σ_6 is Calabi-Yau 3-fold.

In this section we have constructed the holomorphic RW model which is 6-dimensional topological sigma model defined over Calabi-Yau 3-fold with the hyperKähler target. The perturbation theory for this model should give rise to holomorphic invariant of 3-dimensional Calabi-Yau manifolds with the holomorphic volume form.

6 Summary

In this short note we analyzed the RW model and their generalizations within BV formalism. We used the elegant AKSZ-construction which allows to construct the solution of classical master equation from simple geometrical data. The AKSZ treatment of RW model is a bit exotic example since we work with graded symplectic manifold with the additional parameter ("coupling constant") with non-zero grading.

AKSZ-BV framework is very powerful both conceptually and technically and many issues can be systematically addressed within this framework, such as boundary conditions for RW model, the coupling of RW model with Chern-Simons theory etc.

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